Computing definite integrals

Find $\displaystyle\int_{-1}^{2}(2x^2-3)\,dx$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$.

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$.

$x_k = a + k\Delta x = -1 + \frac{3k}{n}$

$f(x_k) = 2x_k^2 - 3 = 2\cdot\left(-1+\frac{3k}{n}\right)^2 - 3$

$$\int_{-1}^{2}(2x^2-3)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(2\cdot\left(-1+\frac{3k}{n}\right)^2-3\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left(2\left(1-\frac{6k}{n}+\frac{9k^2}{n^2}\right)-3\right)$$

$$2-\frac{12k}{n}+\frac{18k^2}{n^2}-3$$

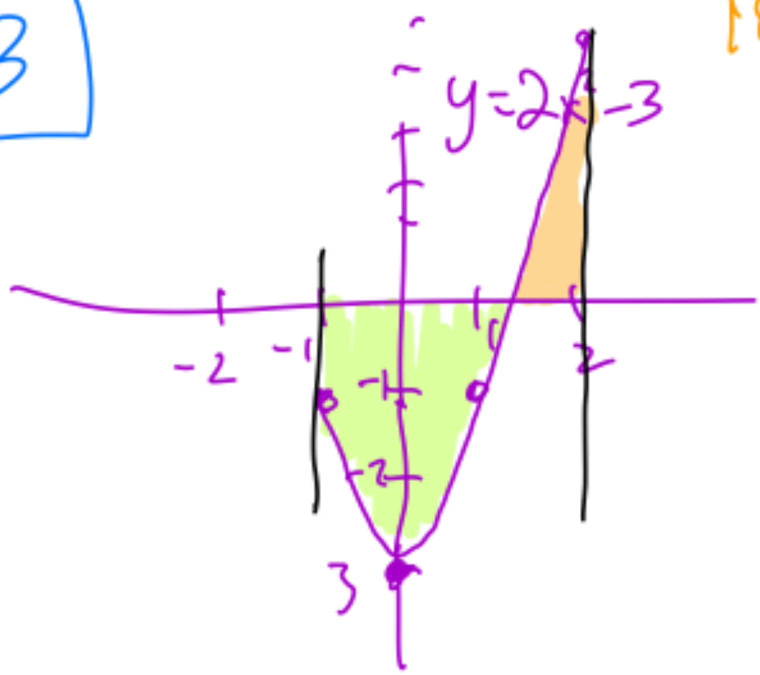
$$= \lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n} -1-\frac{12k}{n}+\frac{18k^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(-1\underbrace{\left(\sum_{k=1}^{n} 1\right)}_{n} - \frac{12}{n}\underbrace{\left(\sum_{k=1}^{n} k\right)}_{\frac{n(n+1)}{2}} + \frac{18}{n^2}\underbrace{\left(\sum_{k=1}^{n} k^2\right)}_{\frac{n(n+1)(2n+1)}{6}}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(-n - \frac{\overset{6}{\cancel{12}}}{\cancel{n}}\,\frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\overset{3}{\cancel{18}}}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}}\right)$$

$$= \lim_{n\to\infty} -3 - \underbrace{\frac{18(n+1)}{n}}_{18} + \underbrace{\frac{9(n+1)(2n+1)}{n^2}}_{18}$$

$$= \boxed{-3}$$



Example Find, using the definition,

$$\int_1^3 2^x\,dx$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$.

$f(x) = 2^x$, $\Delta x = \frac{3-1}{n} = \frac{2}{n}$

$x_k = 1 + k\frac{2}{n} = 1 + \frac{2k}{n}$

$f(x_k) = 2^{1+\frac{2k}{n}} = 2 \cdot 2^{\frac{2k}{n}}$

$$\int_1^3 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 \cdot 2^{\frac{2k}{n}}\right) \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{k=1}^{n} 2^{\frac{2k}{n}} = \lim_{n\to\infty} \frac{4}{n} \left(\sum_{k=1}^{n} \left(2^{\frac{2}{n}}\right)^k\right)$$

geometric $\sum_{k=0}^{B} ar^k = \frac{a(1-r^{B+1})}{1-r}$ $a = 1^{st}$ term, $r =$ ratio.

$B+1 = \#$ of terms

$$= \lim_{n\to\infty} \frac{4}{n} \left(\begin{array}{l} \text{geometric} \\ a = 2^{2/n} = \text{first term in sum} \\ r = 2^{2/n} \\ \#\text{ of terms} = n \end{array} \right)$$

$$= \lim_{n\to\infty} \frac{4}{n} \cdot \left(\frac{a(1-r^{\#})}{1-r} \right)$$

$$= \lim_{n\to\infty} \frac{4}{n} \cdot \left(\frac{2^{2/n}\left(1-\left(2^{2/n}\right)^{n}\right)}{1-\left(2^{2/n}\right)} \right)$$

$$= \lim_{n\to\infty} \frac{4}{n} \frac{\left(2^{2/n}\left(1-2^{2}\right)\right)}{1-2^{2/n}}$$

$$= \lim_{n\to\infty} \frac{4}{n} \cdot \frac{2^{2/n}(-3)}{1-2^{2/n}}$$

$$\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$$

$$= \lim_{n\to\infty} \frac{4}{n} \frac{2^{2/n} \cdot 3}{\left(2^{2/n}-1\right)}$$

$$2^{2/n} \to 2^{0} = 1$$

$$= \lim_{n\to\infty} \frac{12}{n} \frac{}{e^{(\ln 2)\cdot 2/n}-1}$$

$$2^{\frac{2}{n}} = \left(e^{\ln 2}\right)^{\frac{2}{n}} = e^{(\ln 2)\cdot\frac{2}{n}}$$

Geometric series

$$\sum_{k=1}^{n} 2\cdot 4^k = 2\cdot 4^1 \overset{r}{+} 2\cdot 4^2 + \cdots + \cdots + 2\cdot 4^n$$

$a = 1^{st}$ term $= 8$

$r = 4$

# of terms $= n$

$$= \frac{a(1-r^n)}{1-r} = \frac{8(1-4^n)}{1-4}$$

$$= \frac{8(1-4^n)}{-3}$$

$n=2$: $2\cdot 4^1 + 2\cdot 4^2 = 8 + 32 = 40.$

$$\frac{8(1-4^2)}{-3} = \frac{8(1-16)}{-3} = \frac{8(-15)}{-3} = 8\cdot 5$$

$= 40.$ ✓

Geometric

$$\sum_{k=0}^{m} A\cdot p^k = A\cdot p^0 + A\cdot p^1 + \dots + A\cdot p^m$$

$a = 1^{st}$ term $= A$

$r = p$

$\#$ terms $= m+1$

$$= \frac{A\cdot(1-p^{m+1})}{(1-p)}$$

\# terms

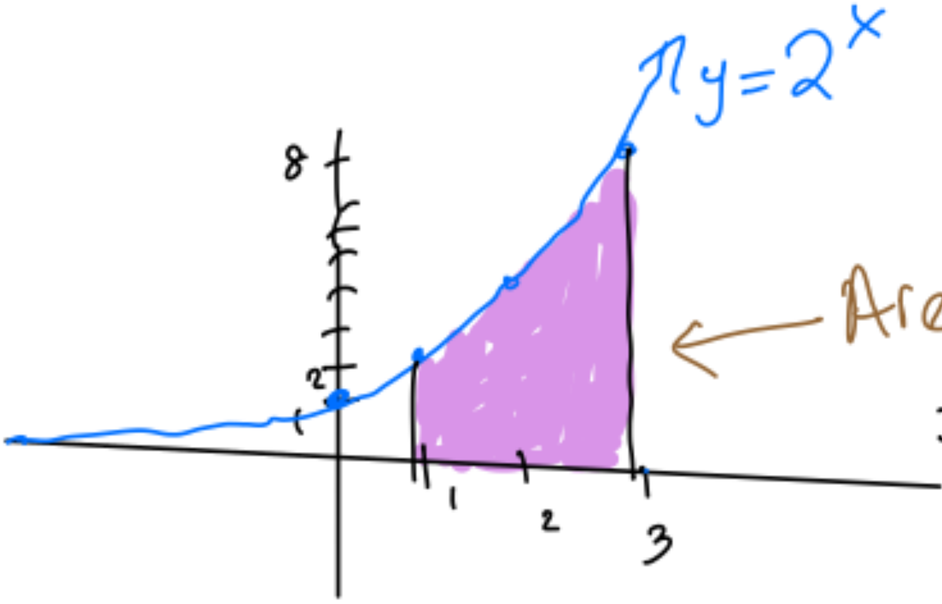
$$= \lim_{n\to\infty} \frac{12}{n} \frac{\left((\ln 2)\frac{2}{n}\right)}{\left(e^{(\ln 2)2/n}-1\right)} \cdot \frac{1}{(\ln 2)\frac{2}{n}}$$

$\to 1$ special limit

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}} \cdot \frac{1}{(\ln 2)\frac{2}{\cancel{n}}}$$

$$= \boxed{\frac{6}{\ln 2}}$$



$\text{Area} = \frac{6}{\ln 2}$

$= \int_1^3 2^x\,dx$.

FUNdamental Theorem of Calculus. (FTC)

If $F'(x)$ is a continuous function on $[a,b]$,
(F is a C^1 function (it's derivative is continuous))

Then $\int_a^b \underbrace{F'(x)}_{\text{derivative of fcn } F}\,dx = F(b) - F(a)$ $\overset{\text{Notation}}{=} F(x)\Big|_a^b$

actual function F.

Example: $\int_{-1}^{2} \underbrace{(2x^2-3)}_{F'(x)}\,dx = F(x)\Big|_a^b = F(b) - F(a)$

$\Rightarrow F(x) = \frac{2}{3}x^3 - 3x$ antiderivative

$= \left(\frac{2}{3}(2)^3 - 3\cdot 2\right) - \left(\frac{2}{3}(-1)^3 - 3(-1)\right)$

$$= \frac{16}{3} - 6 - \left(\frac{-2}{3} + 3\right)$$

$$= \frac{16}{3} - 6 + \frac{2}{3} - 3$$

$$= \frac{16}{3} - \frac{18}{3} + \frac{2}{3} - \frac{9}{3} = \frac{-9}{3} = \boxed{-3}.$$

Example $F(t)$ = displacement (position) at time t

$F'(t)$ = velocity at time t.

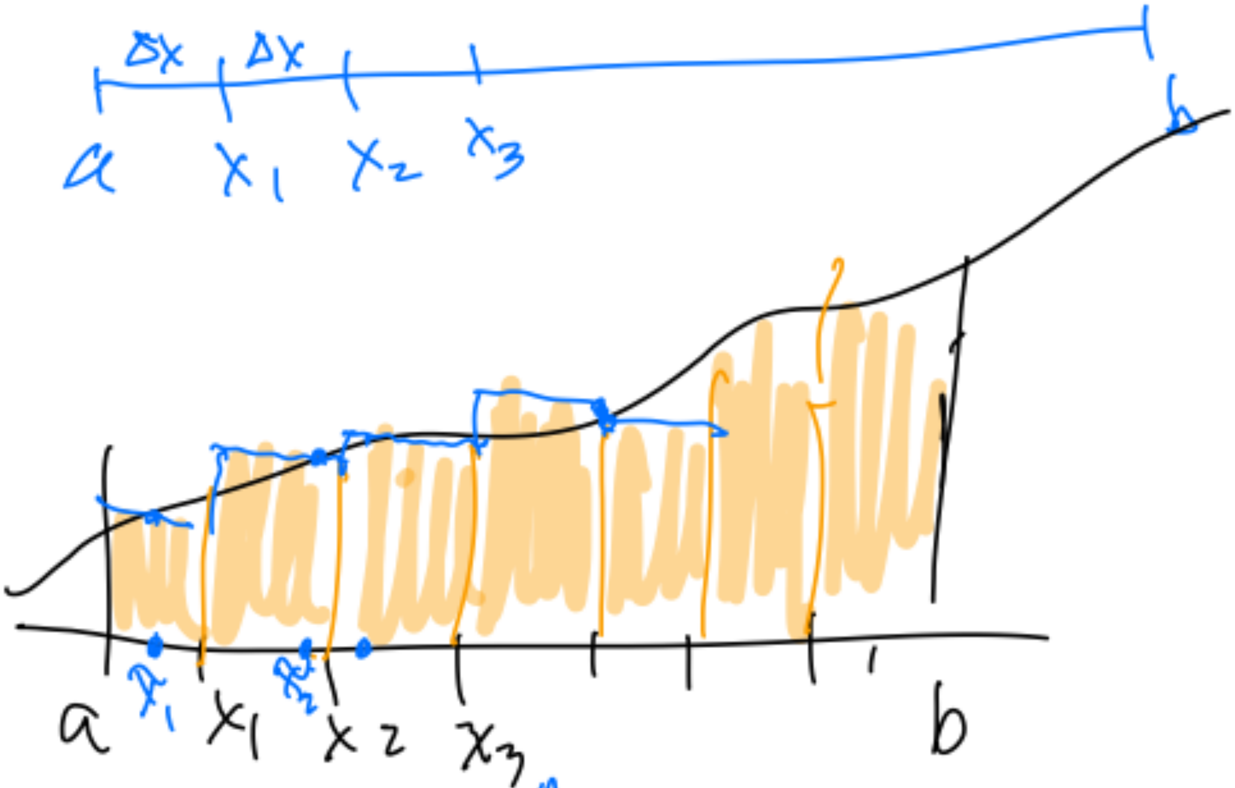
$$\int_a^b F'(t)\,dt = \underbrace{\lim_{n\to\infty} \sum_{k=1}^{n} \underbrace{\underbrace{F'(t_k)}_{\text{velocity}} \underbrace{\Delta t}_{\text{time interval}}}_{\text{displacement}}}_{\text{total displacement}}$$

$$= \underbrace{F(b)}_{\text{position at } t=b} - \underbrace{F(a)}_{\text{position at } t=a} = \underline{\cancel{\text{total}}\ \text{displacement}}$$

Idea of

Proof of FTC:

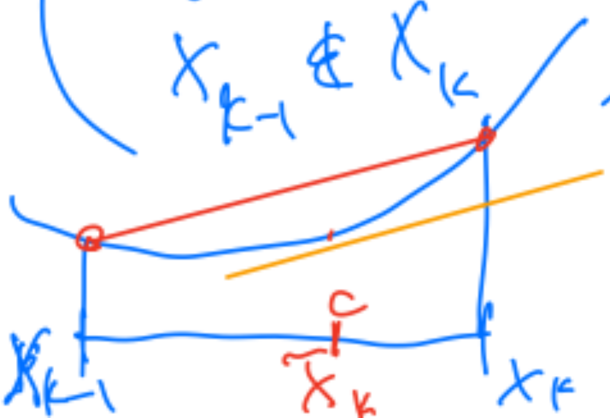
$$\int_a^b F'(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} F'(x_k)\Delta x$$



$$= \lim_{n\to\infty} \sum_{k=1}^{n} F'(\tilde{x}_k)\Delta x$$

(any point between x_{k-1} & x_k)

mean value thm:

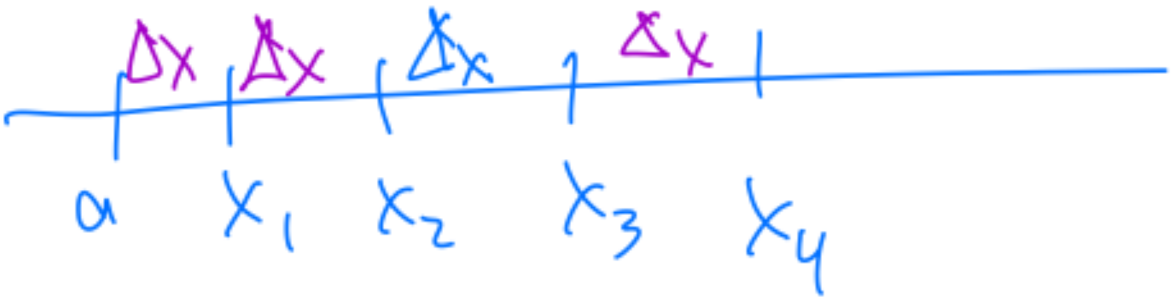


There is a $c \in (x_{k-1}, x_k)$ $\quad c = \tilde{x}_k$

Such that $F'(\underset{\tilde{x}_k}{c}) = \dfrac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} F'(\tilde{x}_k)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{F(x_k) - F(x_{k-1})}{\underbrace{(x_k - x_{k-1})}_{\Delta x}} \Delta x$$



$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(F(x_k) - F(x_{k-1})\right)$$

$$= \lim_{n\to\infty} \left[\cancel{F(x_1)} - F(x_0) + \cancel{F(x_2)} - \cancel{F(x_1)} + \cancel{F(x_3)} - \cancel{F(x_2)} + \cancel{\cdots} + F(x_n) - \cancel{F(x_{n-1})}\right]$$

$$= -F(x_0) + F(x_n)$$

$$x_k = a + k\Delta x$$

$$x_k = a + k\frac{(b-a)}{n}$$

$$\Rightarrow x_0 = a + 0\frac{(b-a)}{n} = a$$

$$x_n = a + n\left(\frac{b-a}{n}\right)$$

$$= a + b - a = b.$$

$$= -F(a) + F(b). \quad \blacksquare$$

$\boxed{\text{Example}}$ $\displaystyle\int_1^3 2^x\,dx$

$\uparrow$ $F'(x)$

If we let $F(x) = \dfrac{2^x}{\ln 2}$?

then $F'(x) = 2^x$.

$$\frac{(2^x)'}{\ln 2} = \frac{(\ln 2)2^x}{\ln 2}$$

$$\left(\frac{2^x}{\ln 2}\right)' = \frac{1}{\ln 2}(2^x)'$$

$$= \frac{1}{\ln 2}(\ln 2)2^x = 2^x$$

$$\int_1^3 2^x\,dx = \left(\frac{2^x}{\ln 2}\right)\Bigg|_1^3$$

$$= \frac{2^3}{\ln 2} - \frac{2^1}{\ln 2} = \frac{8}{\ln 2} - \frac{2}{\ln 2}$$

$$= \frac{6}{\ln 2}$$

Example

$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx = \text{Help!}$$

Only way to calculate is to use definition (or something like it with large n to approximate.)

$$\left(-\cos(x^2)\right)' = \sin(x^2)\cdot 2x$$

Nope.

$$\left(\frac{-\cos(x^2)}{2x}\right)' = \text{quotient rule}$$

Nope.

Celebration of Knowledge Quiz

1. Give $\lim_{n \to \infty}(\text{Your Name})$.

2. Favorite Quote.

3. Find the derivative

a) 2^x

b) $\cos(x)$

c) $\ln(x)$

d) $\arctan(x)$

e) x

f) $\sqrt{x}$

g) x^2

h) x^3

i) x^4.